SADLER UNIT 4 MATHEMATICS METHODS

WORKED SOLUTIONS

Chapter 6 Sample proportions

Exercise 6A

Question 1

Survey 1: $\hat{p} = \dfrac{49}{123} = 0.3984$

Survey 2: $\hat{p} = \dfrac{761}{2348} = 0.3241$

The second sample would be a better estimate as it is larger.

Question 2

Estimate the population proportion by finding the mean of the sample proportions.

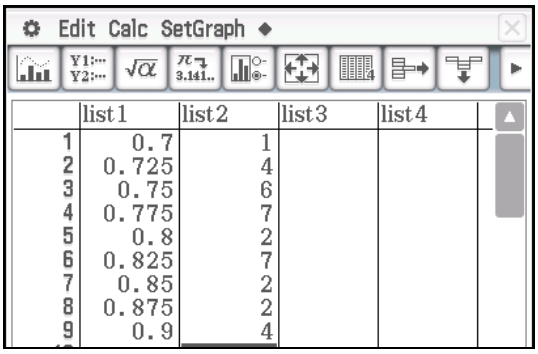
$$p \approx \frac{0.72 + 0.68 + 0.88 + 0.56 + 0.60 + 0.76 + 0.64 + 0.84 + 0.72 + 0.8}{10}$$

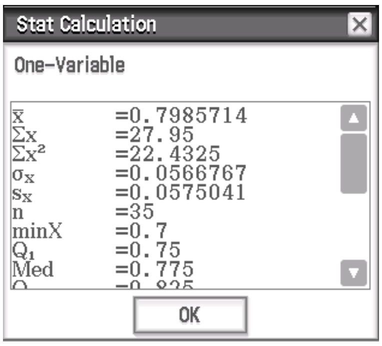
$$\approx 0.72$$

Question 3

a There were 35 (1 + 4 + 6 + 7 + 2 + 7 + 2 + 2 + 4) samples involved.

b Estimate p by finding the mean of the sample proportions:





$p \approx 0.799$

Question 4

a $\hat{p} = \dfrac{72}{320} = 0.225$

b $p = 0.25$ (by geometry of spinner)

c $\bar{x}_{\hat{p}} = p = 0.25$

$\text{s.d.} = \sqrt{\dfrac{0.25 \times 0.75}{320}} = 0.024$

Question 5

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a $p = \dfrac{26}{36} = \dfrac{13}{18} = 0.7\dot{2}$

b $\hat{p} = \dfrac{67}{100} = 0.67$

c

mean of sample proportions $= \dfrac{13}{18} = 0.7\dot{2}$

standard deviation of sample proportions $= \sqrt{\dfrac{\frac{13}{18} \times \frac{5}{18}}{100}} = 0.0448$

d $\dfrac{0.67 - \frac{13}{18}}{0.0448} = -1.1657$

Approximately 1.166 standard deviations below the population proportion

Question 6

a $p = 0.84$

b $\hat{p} = \frac{147}{240} = 0.6125$

c The proportions are very different. Analysis shows the following:

Given $p = 0.84, n = 240$

standard deviation of sample proportions

$= \sqrt{\frac{0.84 \times 0.16}{240}}$

$= 0.0237$

i.e. $\hat{p} \sim \mathrm{N}(0.84, 0.0237)$

$\frac{0.6125 - 0.84}{0.0237} = -9.6$

Our sample's proportion is 9.6 standard deviations below the population proportion, *p* which is extremely unlikely, suggesting the sample is not representative of the population.

Question 7

a

$p \approx 0.1$

$\hat{p} = \frac{112}{1000} = 0.112$

Given $p = 0.1,\ n = 1000$,

standard deviation of sample proportions

$= \sqrt{\frac{0.1 \times 0.9}{1000}}$

$= 0.0095$

We would expect $\hat{p} \sim \mathrm{N}(0.1, 0.0095)$

$\frac{0.1 - 0.112}{0.0095} = 1.26$

$\hat{p}$ is 1.26 standard deviations above p

Hence the school appears to be reasonable representative of the population.

b $\hat{p} = 0.35$

The survey does not give *n*, so we are unable to determine the expected standard deviation,

but the significant difference between p and $\hat{p}$ suggests this sample is not representative of the population at large.

Question 8

a $\hat{p} = \frac{461}{1247} = 0.3697$

b standard deviation of sample proportions

$= \sqrt{\frac{\frac{461}{1247} \times \left(1 - \frac{461}{1247}\right)}{1247}}$

$= 0.0137$

Question 9

a $\hat{p} = \dfrac{143}{248} = 0.577$

b standard deviation of sample proportions

$$= \sqrt{\frac{\frac{143}{248} \times \left(1 - \frac{143}{248}\right)}{248}}$$

$$= 0.0314$$

Question 10

a The huge variation in sample sizes means it would be wise to only consider the larger samples as they give better estimates for proportions.

b Convert the proportions to actual numbers of people having high blood pressure in each sample, and then treat the data as one large sample. See textbook answer for table of solutions.

$$\hat{p} = \frac{(0.5 \times 8) + (0.1 \times 10) + ... + (0.8 \times 10) + 1}{8 + 10 + 50 + ... + 25 + 10 + 1}$$

$$= \frac{193}{756}$$

$$= 0.255$$

Question 11

$p = 0.1, n = 200, \hat{p} = \frac{35}{200} = 0.175$

standard deviation of sample proportions

$= \sqrt{\frac{0.1 \times 0.9}{200}}$

$= 0.0212$

$\frac{0.175 - 0.1}{0.0212} = -3.54$

The sample proportion, $\hat{p}$ is 3.54 standard deviations below the mean, falling into the extremely unlikely category.

Question 12

To be confident a sample distribution will approximate to the normal distribution we need two conditions to be satisfied, after besides a sample of more than 30.

$np \geq 10$ and $n(1-p) \geq 10$.

In graph one, $np = 50 \times 0.5 = 25, n(1-p) = 50 \times 0.5 = 25$ so both the conditions are fulfilled and we could expect a normal distribution of the sample proportions.

In graph two, $np = 50 \times 0.05 = 2.5, n(1-p) = 50 \times 0.95 = 47.5$ so while the sample size is larger, is does not meet the necessary conditions.

Question 13

$n \geq 30$ $\quad np = 100 \times 0.5 = 50 \geq 10$ $\quad n(1-p)100 \times 0.5 = 50 \geq 10$

Given $p = 0.5, n = 100$

standard deviation of sample proportions

$= \sqrt{\frac{0.5 \times 0.5}{100}}$

$= 0.05$

We would expect the proportions to be normally distributed with a mean of 0.5
and standard deviation of 0.05.
68% of the samples should have proportions between 0.5 ± 0.05,
$0.45 \leq \hat{p} \leq 0.55$.

Question 14

The predicted value of $p =$ 0.52.

$\hat{p} = \frac{81}{200} = 0.405$

standard deviation of sample proportions

$= \sqrt{\frac{0.405 \times (1 - 0.405)}{200}}$

$= 0.0347$

$\frac{0.405 - 0.52}{0.0347} = -3.31$

The sample proportion is 3.3 standard deviations from the expected mean, so it is unlikely a sample which is representative of the population would return this result.

Question 15

$p = 0.24, n = 800$

standard deviation of sample proportions

$= \sqrt{\frac{0.24 \times 0.76}{800}}$

$= 0.0151$

$P(-k < Z < k) = 0.90$

$k = 1.6449$

90% of scores in a normal distribution occur with 1.6449 standard deviations of the mean

$-1.6449 = \frac{x - 0.24}{0.0151}$

$x = 0.2152$

$1.6449 = \frac{x - 0.24}{0.0151}$

$x = 0.2648$

It would be reasonable to expect 90% of the samples to have proportions between 21.5% and 26.5%.

Exercise 6B

Question 1

$n = 1200$

$\hat{p} = \dfrac{450}{1200} = 0.375$

standard deviation of sample proportions

$= \sqrt{\dfrac{0.375 \times (1 - 0.375)}{1200}}$

$= 0.014$

Critical value for 95% confidence interval : 1.960

$0.376 - 1.960 \times 0.014 = 0.34756$

$0.376 + 1.960 \times 0.014 = 0.40244$

The 95% CI is 0.3476 to 0.4024

We could expect 95% of the 95% confidence intervals to contain the population proportion.

Hence with 95% confidence we estimate between 34.76% and 40.24% of people are in favour of national service.

Question 2

$n = 800$

$\hat{p} = \dfrac{680}{800} = 0.85$

standard deviation of sample proportions

$= \sqrt{\dfrac{0.85 \times (1 - 0.85)}{800}}$

$= 0.0126$

Critical value for 90% confidence interval: 1.645

$$0.85 \pm 1.645 \times \sqrt{\dfrac{0.85 \times 0.15}{800}} \Rightarrow 0.8292 \text{ to } 0.8708$$

We could expect 90% of the 90% confidence intervals to contain the population proportion.

Hence with 90% confidence we estimate between 82.92% and 87.08% of people were satisfied or very satisfied with the service received.

Question 3

$n = 250$

$\hat{p} = \dfrac{190}{250} = 0.76$

standard deviation of sample proportions

$= \sqrt{\dfrac{0.76 \times 0.24}{250}}$

$= 0.0270$

Critical value for 99% confidence interval: 2.576

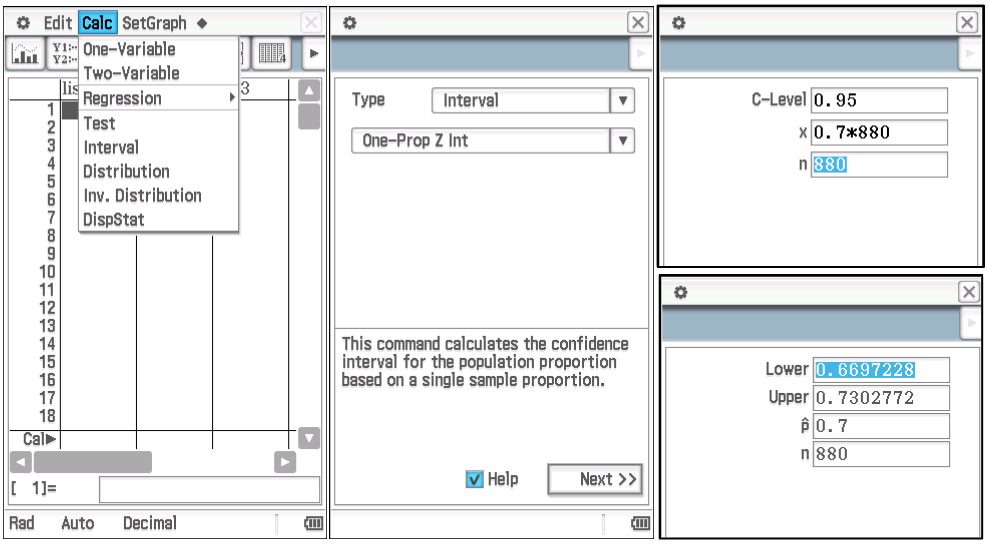
$$0.76 \pm 2.576 \times \sqrt{\frac{0.76 \times 0.24}{250}} \Rightarrow 0.6904 \text{ to } 0.8296$$

We could expect 99% of the 99% confidence intervals to contain the population proportion.

Hence with 99% confidence we estimate between 69.04% and 82.96% of people agreed with the rule changes.

Question 4

By ClassPad



$0.6697 \text{ to } 0.7303 \Rightarrow 70\% \pm 3\%$

Question 5

$n = 2000$
$\hat{p} = 0.45$

Critical value for 90% confidence interval : 1.645

$$0.45 \pm 1.645 \times \sqrt{\frac{0.45 \times 0.55}{2000}} \Rightarrow 0.4317 \text{ to } 0.4683$$
$$90\%\text{CI} : 43.2\% \text{ to } 46.8\%$$

70% is a long way outside the confidence interval suggesting the sample of 200 is not representative of the larger community. (See text for a fuller response)

Question 6

$n = 1000$
$p = 0.28$

Critical value for 95% confidence interval : 1.960

$$0.28 \pm 1.960\sqrt{\frac{0.28 \times 0.72}{1000}} \Rightarrow 0.2522 \text{ to } 0.3078$$

We could expect 95% of the 95% confidence intervals to contain the population proportion. Hence with 95% confidence we estimate between 25.2% and 30.8% of seeds in a sample this size would fail to germinate. This batch fits with this estimate.

Question 7

Critical value for 95% confidence interval : 1.960

$$1.960\sqrt{\frac{0.45 \times 0.55}{2000}} = 0.0218$$

Question 8

By definition a 99% confidence interval is wider and has a greater margin of error for samples of the same size and sample proportions.

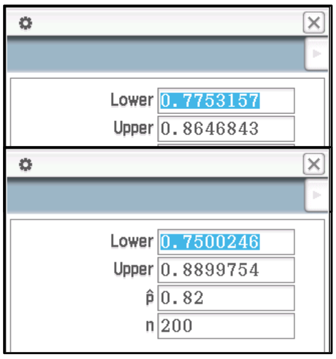
Question 9

a $\frac{36}{200} = 0.18$

Porportion of acceptable components = 0.82

b With 90% confidence we estimate that at the time the sample was taken, between 77.5% and 86.5% of components made by this machine were of an acceptable standard.

c With 99% confidence we estimate that at the time the sample was taken, between 75.0% and 89.0% of components made by this machine were of an acceptable standard.



Question 10

Critical value for 95% confidence interval : 1.960

$$0.065 = 1.960 \times \sqrt{\frac{0.5 \times 0.5}{n}}$$

$n = 227.31$

To nearest person and to be inside the margin of error,

a sample size of 228 is required.

$$\begin{pmatrix} \text{If } n = 227, \text{ margin of error is } 0.065\ 045 \\ \text{If } n = 228, \text{ margin of error is } 0.064\ 902 \end{pmatrix}$$

Question 11

Critical value for 90% confidence interval: 1.645

$$0.03 = 1.645 \times \sqrt{\frac{0.5 \times 0.5}{n}}$$

$n = 751.67$

To nearest person and to be inside the margin of error,

a sample size of 752 is required.

Question 12

Critical value for 95% confidence interval: 1.960

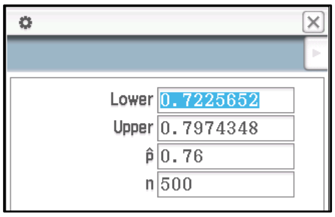
0.60 to 0.70 gives margin of error of 0.05.

$$0.05 = 1.960 \times \sqrt{\frac{0.65 \times 0.35}{n}}$$

$n = 349.59$

To nearest person and to be inside the margin of error,
a sample size of 350 is required.

Question 13



We can be 95% confident that of all Australian males between the ages of 20 and 30, between 72% and 80% are taller than their father.

To be 99% confident we require a larger interval to be more confident that the population proportion will lie in our interval.

Question 14

$0.241 - 0.060 = 0.181$

$0.241 + 0.060 = 0.301$

We can be 90% confident that the proportion of Australians having the particular attribute lies between 18% and 30%.

Question 15

a $\dfrac{168}{480} = 0.35$

b standard deviation of sample proportions

$$= \sqrt{\frac{0.35 \times 0.65}{480}}$$
$$= 0.0218$$

c $0.35 \pm 1.960 \times \sqrt{\dfrac{0.35 \times 0.65}{480}} \Rightarrow 0.3073 \text{ to } 0.3927$

We could expect 95% of the 95% confidence intervals to contain the population proportion. Hence with 95% confidence we estimate between 30.7% and 39.3% of Year 12 Australian school students say they intend to proceed to University in the following year.

d $0.03 = 1.960 \times \sqrt{\dfrac{0.35 \times 0.65}{n}}$

$$n = 971.1$$

A sample of 972 or more is required.

Question 1

a $\log_x 64 = 6$

$x^6 = 64$

$x = 2$

b $\log_2 x = 3$

$2^3 = x$

$x = 8$

c $\log_{10} x = 2$

$x = 10^2$

$x = 100$

d $x = 1$

Question 2

a $\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$

$= \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3\sqrt{x}}{2}$

b $\frac{d}{dx}(4x^5 + \log_e x)$

$= 20x^4 + \frac{1}{x}$

c $\frac{d}{dx}(7\ln x)$

$= 7 \times \frac{1}{x}$

$= \frac{7}{x}$

d $\frac{d}{dx}\left(\ln(5x^3 - 6x)\right)$

$= \frac{15x^2 - 6}{5x^3 - 6x}$

Question 3

a $\dfrac{dy}{dx} = \dfrac{5}{5x-1}$

b $\dfrac{dy}{dx} = \dfrac{4x^3}{x^4+1}$

c $y = \ln(x+1) + \ln(x-1)$

$$\frac{dy}{dx} = \frac{1}{x+1} + \frac{1}{x-1}$$
$$= \frac{x-1+x+1}{(x+1)(x-1)}$$
$$= \frac{2x}{x^2-1}$$

Question 4

$y = 3 - \ln x$

$\dfrac{dy}{dx} = -\dfrac{1}{x}$

When $x = e$,

$\dfrac{dy}{dx} = -\dfrac{1}{e}$

Equation of tangent

$y = -\dfrac{1}{e}x + c$

Using $(e,\ 2)$

$2 = -\dfrac{1}{e}(e) + c$

$2 = -1 + c$

$c = 3$

$\therefore\ y = -\dfrac{1}{e}x + 3$

$y = 3 - \dfrac{x}{e}$

Question 5

a $p = 0.25$

b $\hat{p} = \dfrac{46}{160} = 0.2875$

c mean $= 0.25$

standard deviation of sample proportions

$$= \sqrt{\frac{0.25 \times 0.75}{160}}$$

$= 0.0342 \quad (4\text{dp})$

d $\dfrac{0.2875 - 0.25}{0.0342} = 1.096$

Approximately 1.1 standard deviations above *p*.

Question 6

a $k \times 8 = 1$

$$k = \frac{1}{8}$$

b $\mathrm{P}(X \geq 4) = \dfrac{3}{4}$

c $\mathrm{P}(X < 8) = \dfrac{3}{4}$

d $\mathrm{P}(X > 3 \mid X < 7)$

$$= \frac{\mathrm{P}(3 < X < 7)}{\mathrm{P}(X < 7)}$$

$$= \frac{1}{2} \div \frac{5}{8}$$

$$= \frac{4}{5}$$

Question 7

$$\int_0^5 0.5e^{-0.5x}dx$$

$= 0.9179$

	H	T
H	HH	HT
T	TH	TT

Question 8

$$f'(x)=\frac{x+5}{x}=1+\frac{5}{x}$$

$$f''(x)=-\frac{5}{x^2}$$

$$f(x)=x+5\ln x+c$$

$$\begin{aligned} f(1)=1+5\ln 1+c&=5\\ 5\ln 1+c&=4\\ c&=4-5\ln 1 \end{aligned}$$

$$\begin{aligned} \therefore\ f(x)&=x+5\ln x+4-5\ln 1\\ &=x+5(\ln x-\ln 1)+4\\ &=x+5\ln\left(\frac{x}{1}\right)+4\\ &=x\ \ +5\ln x+4 \end{aligned}$$

Question 9

a $\frac{1}{k}\int_1^5\left(\frac{1}{x}\right)dx$

$$=\frac{1}{k}\left[\ln x\right]_1^5$$

$$=\frac{1}{k}\left(\ln 5-\ln 1\right)$$

$$=\frac{1}{k}\ln 5$$

$$1=\frac{1}{k}\ln 5$$

$$k=\ln 5$$

b $\mathrm{P}(2\le X\le 4)$

$$=\int_2^4\left(\frac{1}{x\ln 5}\right)dx$$

$$=\frac{1}{\ln 5}\int_2^4\frac{1}{x}dx$$

$$=\frac{1}{\ln 5}\left[\ln x\right]_2^4$$

$$=\frac{1}{\ln 5}\left[\ln 4-\ln 2\right]$$

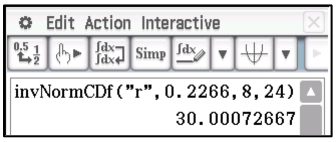
$$=\frac{1}{\ln 5}\left[\ln\left(\frac{4}{2}\right)\right]$$

$$=\frac{\ln 2}{\ln 5}$$

Question 10

$$P(X>k)=0.2266$$

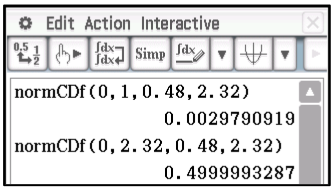
$$k=30$$



Question 11

Let T represent times on test.

$T \sim N(2.32,\ 0.48^2)$



a $\quad P(T < 1) = 0.0030$

b $\quad P(T < 1 \mid T < 2.32)$

$$= \frac{P(T < 1)}{P(T < 2.32)}$$

$$= \frac{0.003}{0.5}$$

$$= 0.006$$

Question 12

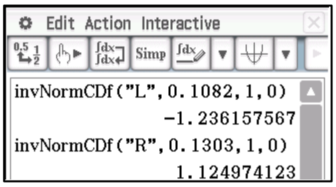
a $\quad 1 - 0.1082 - 0.1303$

$$= 0.76$$

b $\quad P(X < 25) = 0.1082$

25 is 1.2362 standard deviations below the mean.

$$-1.2362 = \frac{25 - \mu}{\sigma} \quad \rightarrow \quad \text{Equation 1}$$



42 is 1.1250 standard deviations above the mean.

$$1.1250 = \frac{42 - \mu}{\sigma} \quad \rightarrow \quad \text{Equation 2}$$

Solving simultaneously

$\mu = 33.90$

$\sigma = 7.20$

Question 13

$\hat{p} = 0.521$

$p = 0.5$

standard deviation of sample proportions

$= \sqrt{\dfrac{0.521 \times (1 - 0.521)}{1000}}$

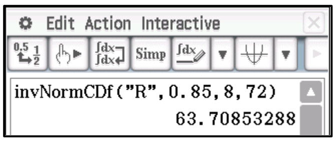
$= 0.0158$

We would expect the sample proportions to be distributed normally with a mean of 0.5 and a standard deviation of 0.0158.

$\dfrac{0.521 - 0.5}{0.0158} = 1.33$

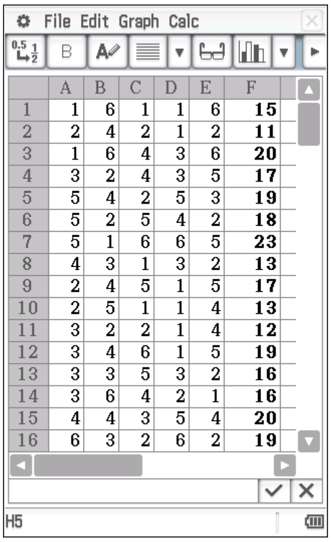
$\hat{p}$ is 1.3 standard deviations above the mean which does not suggest the coin is unfair.

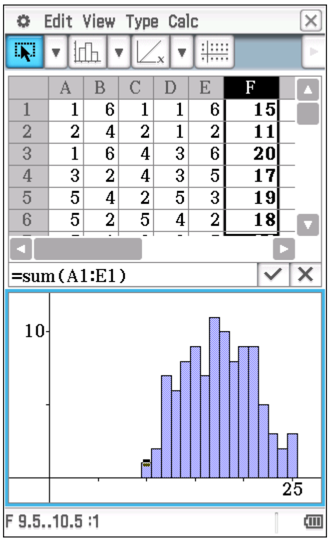
Question 14

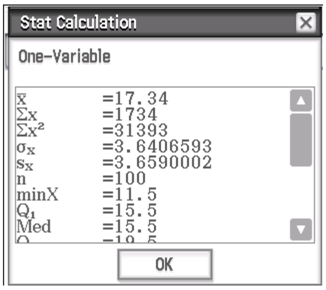


The pass mark is 63.5%.

Question 15





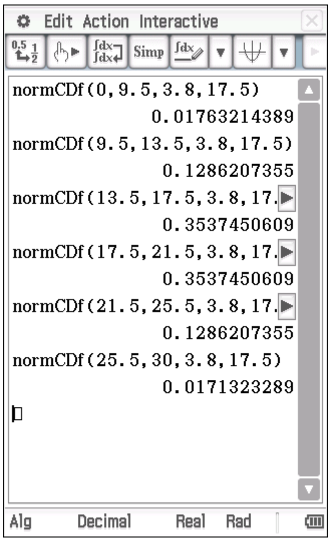


Score	≤ 9	$10 \to 13$	$14 \to 17$	$18 \to 21$	$22 \to 25$	≥ 26
Frequency	0	16	35	36	13	0

Let X represent the sum of the uppermost faces of the five dice.

The adjustments for continuity and subsequent theoretical distribution suggested:

Score	$X \le 9.$	$9.5 \le X \le 13.5$	$13.5 \le X \le 17.5$	$17.5 \le X \le 21.5$	$21.5 \le X \le 25.5$	$X \ge 25.5$
Frequency	2	13	36	36	13	2



The suggested model fits well on initial inspection. The data is symmetrical but the simulated data has a smaller range and therefore standard deviation. This is confirmed by the summary statistics from the simulated grouped data (mean of 17.34 and a standard deviation of 3.6)

We would expect approximately 68% of the data to fall within 1 standard deviation of the mean. The simulated data has 71% falling with between 14 and 21 (1 standard deviation either side of the mean 13.7 to 20.9) suggests the model suggested is suitable.

Using the data to suggest scores occurring two standard deviations either side of the mean proposes a group of 10.14 to 24.5. If we consider the groups 10 to 25 from our data, we find 100% our data compared to an expected 95%. Consideration of the ungrouped data shows 97% of the scores fall within 2 standard deviations of the mean.

A model with a slightly lower standard deviation may fit a little better but overall the model is suitable.

Question 16

The sample proportion for the survey of older Australians: $\hat{p} = \frac{20}{500} = 4\%$

The sample proportion for the survey of younger Australians: $\hat{p} = \frac{336}{800} = 42\%$

Both are considerably different from the sample proportion of Australian adults.

See text book answer for a full response.

Question 17

$p = 0.22$

$\hat{p} = 0.2, n = 400$

standard deviation of sample proportions

$$\sqrt{\frac{0.22 \times 0.78}{400}} = 0.0207$$

The sample proportion is approximately 1 standard deviation from the mean which is a likely variation.

If the sample had been from 4000 people

standard deviation of sample proportions

$$\sqrt{\frac{0.22 \times 0.78}{4000}} = 0.0065$$

$$\frac{0.2 - 0.22}{0.0065} = -3.08$$

The sample proportion is more than 3 standard deviation from the mean which is an unlikely variation.

See text book answer for a full response.

Question 18

$n = 100,\ p = 0.15$

standard deviation of sample proportions

$$\sqrt{\frac{0.15 \times 0.85}{100}} = 0.0036$$

We expect the sample proportions to be normally distributed with a mean of 0.15 and a standard deviation of 0.036.

$$\frac{0.17 - 0.15}{0.036} = 0.56$$

A sample with 17 faulty batteries is a likely and acceptable result for the given parameters.

Joe can assume his batch came from the population described.

See text book answer for a full response.

